Some Generalized BRS Transformations. I
The Pure Yang-Mills Case

Paul Federbush
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1109
(pfed@math.lsa.umich.edu)

Abstract

Some generalized BRS transformations are developed for the pure Yang-Mills theory, and a form of quantum gravity. Unlike the usual BRS transformations: these are nonlocal; may be infinite formal power series in the gauge fields; and do not leave the actio n invariant, but only the product e^{-S} with the Jacobian. Similar constructions should exist for many other field theory situations.

I. Introduction

Since the development of BRS transformations for the Yang-Mills theory, [1], they have played a major role in theoretical applications, such as to the study of renormalization and unitarity. BRS transformations have also been given for quantum gravity [2] ,[3],[4], and applied to study the renormalizability of higher derivative quantum gravity, [4]. Our interest was to develop a BRS transformation for a particular formulation of quantum gravity in a natural gauge to the theory, [5]. This led us to develop the generalized BRS transformations of this paper, and to apply them to the pure Yang-Mills theory. The Yang-Mills setting is a simpler arena to present the basic ideas, and hopefully generalized BRS transformations may have application to the Yang-Mills theory. There has been study of some aspects of the Yang-Mills theory by other generalizations of the BRS symmetry,[6].

For the pure Yang-Mills we write the action as follows:

$$S = \int \mathrm{Tr}[\alpha \mathrm{F}^2_{\mu\nu} + \frac{\beta}{2}(\partial_\mu \mathrm{A}_\mu)^2 + \gamma \bar{\mathrm{c}}_\mathrm{i}(\partial^\mathrm{L}_\mu \mathrm{L}_\mathrm{i})(\partial_\mu \mathrm{L}_\mathrm{j} + [\mathrm{A}_\mu, \mathrm{L}_\mathrm{j}])\mathrm{c}_\mathrm{j}]. \tag{1}$$

The superscript L indicates differentiation is to the left, and c_i is the ghost field. Sum over repeated indices will always be understood, except where otherwise indicated.

The BRS transformations are then:

$$A_\mu(x) \to A_\mu(x) + (\partial_\mu c_j(x))L_j\lambda + [A_\mu(x), L_j]c_j(x)\lambda. \tag{2}$$

$$\bar{c}_j(x) \to \bar{c}_j(x) - (\frac{\beta}{\gamma})\partial_\mu A^\mu_j(x)\lambda. \tag{3}$$

$$c_j(x) \to c_j(x) + \frac{1}{2}s_{j\kappa\ell}c_\kappa(x)c_\ell(x)\lambda. \tag{4}$$

We work in Euclidean space, and the L_i are orthonormal in the trace inner product.

These transformations leave the action invariant and have (super -) Jacobian 1 (of course working to linear order in λ). The structure constants satisfy:

$$s_{ij\kappa} = \mathrm{Tr}(L_i[L_j, L_\kappa]). \tag{5}$$

II <u>Generalized BRS Transformations for the Pure Yang-Mills</u>

In contrast to (2),(3),(4) the generalized BRS transformations for the pure Yang-Mills theory will involve a rather arbitrary formal gauge transformation and are given as:

$$\begin{aligned} A_\mu(x) \to A_\mu(x) \quad &+ \quad \frac{\partial}{\partial x^\mu}[c_j(x) + \int_y F_j(x,y)c_j(y)]L_j\lambda \\ &+ \quad [A_\mu(x), L_j][c_j(x) + \int_y F_j(x,y)c_j(y)]\lambda \end{aligned} \tag{6}$$

$$\bar{c}_i(x) \to \bar{c}_i(x) - (\frac{\beta}{\gamma})(\frac{\partial}{\partial x^\mu}A_i^\mu(x))\lambda - (\frac{\beta}{\gamma})G_i(x)\lambda \tag{7}$$

$$c_j(x) \to c_j(x) + \frac{1}{2}s_{j\kappa\ell}c_\kappa(x)c_\ell(x)\lambda + \int_y\int_z Z_{j\kappa\ell}(x,y,z)c_\kappa(y)c_\ell(z)\lambda \tag{8}$$

Here $F_j(x,y)$ is an essentially arbitrary formal power series in the A_μ field with the lowest order term of degree 1.

$$F_j(x,y) = F_j^1(x,y) + F_j^2(x,y) + ... \tag{9}$$

$F_j^i(x,y)$ is of degree i. F^1, say, is of form:

$$F_j^1(x,y) = \int_z f_j^{\mu i}(x,y,z,)A_\mu^i(z) \tag{10}$$

where

$$A_\mu(x) = \Sigma_i A^i_\mu(x) L_i \tag{11}$$

The G_i and $Z_{j\kappa\ell}$ are determined as formal power series in the A_μ, inductively by degree, as will be specified below. If $F_j \equiv 0$ one gets the usual BRS transformation. If to order one in λ we write:

$$S \to S + \Delta S \lambda \tag{12}$$

$$J = 1 + \Delta J \lambda \tag{13}$$

Where J is the Jacobian of the transformation (6) –(8), then we require:

$$\Delta S - \Delta J = 0 \tag{14}$$

which ensures invariance of $\int e^{-S}$ (i.e. invariance of e^{-S} times integration measure density).

We write

$$\Delta S = \Delta S_1 + \Delta S_2 \tag{15}$$

$$\Delta J = \Delta J_1 + \Delta J_2 \tag{16}$$

where the subscripts 1 and 2 split the expressions into terms linear and quadratic in $c_i(x)$. Eq. (14) becomes two equations:

$$\Delta S_1 - \Delta J_1 = 0 \tag{17}$$

$$\Delta S_2 - \Delta J_2 = 0 \tag{18}$$

It is easy to see:

$$\Delta J_2 = 0 \tag{19}$$

The equations (18)–(19) are just:

$$\Delta S_2 = 0 \tag{20}$$

which by a simple calculation holds for Z satisfying:

$$
\begin{aligned}
\Delta_x Z_{i\kappa\ell}(x,y,z) &+ \frac{\partial}{\partial x^\mu}(A^\nu_\mu(x) Z_{s\kappa\ell}(x,y,z)) s_{irs} \\
&- \frac{\partial}{\partial x^\mu}[(\frac{\partial}{\partial x^\mu} F_\kappa(x,y))\delta(z-x)] s_{i\kappa\ell} \\
&- \frac{\partial}{\partial x^\mu}[A^s_\mu(x) F_\kappa(x,y)\delta(z-x)] s_{rs\kappa} s_{ir\ell} = 0
\end{aligned}
\tag{21}
$$

$\delta(z-x)$ is a four dimensional delta function. Equation (21) may be solved inductively in degree for Z a formal power series in the fields $A_\mu(x)$, similar to $F_\kappa(x,y)$.
In equation (21) indices i, κ, ℓ are never summed!

With $F_\kappa(x,y)$ given, and $Z_{i\kappa\ell}(x,y,z)$ now determined, $G_i(x)$ is obtained from equation (17), completely specifying the generalized BRS transformation (6)–(8). Similarly to the derivation of $Z_{i\kappa\ell}(x,y,z)$, equation (17) holds if $G_i(x)$ satisfies;

$$
\begin{aligned}
\Delta_y G_i(y) &- (\frac{\partial}{\partial y^\mu} G_r(y)) A^\mu_j(y) s_{rji} + \int_x A^\nu_i(x)_{,\mu,\nu} \frac{\partial}{\partial x^\mu} F_i(x,y) \\
&+ \int_x A^\nu_{r,\mu,\nu}(x) A^\mu_j(x) F_i(x,y) s_{rji} + \frac{1}{\beta}\int_x \frac{\delta}{\delta A^\mu_i(x)} \frac{\partial}{\partial x^\mu} F_i(x,y) \\
&+ \frac{1}{\beta}\int_x \frac{\delta}{\delta A^\mu_r(x)} \left\{ A^j_\mu(x) F_i(x,y) \right\} s_{rji} \\
&- \frac{1}{\beta}\int_x [Z_{jji}(x,x,y) - Z_{jij}(x,y,x)] = 0.
\end{aligned}
\tag{22}
$$

This equation may be solved inductively as a formal power series in $A_\mu(x)$ for G_i.The familiar notation for functional derivative has been used, and commas indicate partial derivatives. Index i is never summed over! The last term is eq.(22) is delicate to calculate... the nitty-gritty yet awaits a proper exegesis.

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